Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 3, pp. 440–448, 2004

MULTIGRID MODELS OF COMPOSITE MATERIALS OF IRREGULAR STRUCTURE WITH A SMALL FILLING RATIO

A. D. Matveev

UDC 539.3

This paper considers composites consisting of a set of typical composite multigrid finite elements whose structures are regular and different. Mean local errors are proposed for multigrid modeling of composites.

Key words: *composites, irregular structure, multigrid finite element, mean local errors.*

Introduction. Composites have been usually studied using micro- and macromodels [1]. In macromodels, a composites is considered a homogeneous body with certain (fictitious) elastic moduli. Furthermore, the deformation of composites is described invoking various hypotheses, depending on composite structure. These hypotheses impose certain constraints on displacements, strains, and stresses, which introduces an inherent error into the solutions. Constructing solutions for a composite macromodel reduces to finding fictitious elastic moduli of the composite, which is a difficult problem. Serious difficulties arise when macromodels are used to analyze composites of irregular structure with a small filling ratio. Micromodels provide an adequate description of the behavior of composites. However, finite-element (basic) composite models constructed using the microapproach have large dimensions [1]. The use of superelements to decrease the dimension of these models is not effective [2].

In the present paper, composites of irregular structure with a small filling ratio are analyzed by multigrid modeling, which reduces to constructing a multigrid discrete model on the basic composite model. This model consists of composite multigrid finite elements (CMFEs) [3, 4]. To design an m-grid composite finite element (FE), one uses m nested grids. The finest grid is generated by basic partition taking into account the CMFE structure, and the remaining $m-1$ grids are determined on its boundary. Construction of CMFEs reduces to eliminating all nodal unknowns in the basic partition inside the region and most of the unknowns on the boundary.

An advantage of multigrid modeling is that multigrid models take into account composite structure and the model dimension is much smaller than that of basic composite models and, hence, finite-element implementation for multigrid models requires much less computer time and memory than that for basic models.

We consider composites consisting of typical square two-grid finite elements of the same size which have identical fine and coarse grids. The composite structures of typical CMFEs are assumed to be regular and different. Calculations show that the error of grid solutions is a function of coordinates. It is therefore reasonable to analyze the solutions by using the mean local errors in grid displacements and stresses in relatively small subregions of the composite. We describe procedures that allow one to construct two-grid models of composites so that the mean local errors in grid displacements or equivalent stresses in indicated subregions are smaller than a certain specified value. An example of calculations is given and calculation results are analyzed.

1. Composite Multigrid Finite Elements. We consider the main principles of designing CMFEs using as an example a composite five-grid rectangular finite element (RFE) $ABCD$ with dimensions $a\times b$ (Fig. 1) subjected to plane stresses. In Fig. 1, the RFE comprises rigid flat fibers of width h (dashed) and a $3h\times 2h$ particle (dashed). We assume that the constraints between the components of the composite RFE are ideal and the displacements, stresses, and strains of these components obey Hooke's law and Cauchy's relations [5]. The basic partition of the RFE, which consists of first-order square finite elements S^h with side h, takes into account its structure and

Institute of Computational Modeling, Krasnoyarsk 660036. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 3, pp. 161–171, May–June, 2004. Original article submitted October 8, 2002; revision submitted October 13, 2003.

 0021-8944/04/4503-0440

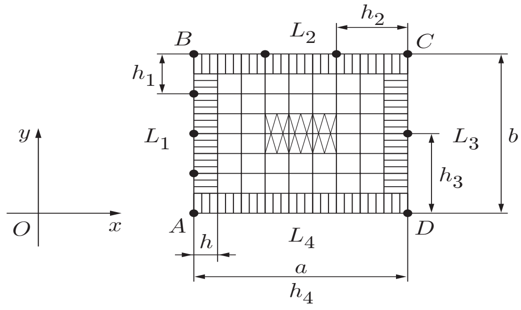

Fig. 1. Composite five-grid RFE.

generates a fine grid S_h with size h, whose nodal unknowns are the displacements u and v. On the grid S_h, we construct a rectangular superelement [2], whose potential energy Π_s^e is given by

$$\Pi_s^e(\boldsymbol{v}_s^e) = (1/2)(\boldsymbol{v}_s^e)^{\mathrm{t}} K_s^e \boldsymbol{v}_s^e - (\boldsymbol{v}_s^e)^{\mathrm{t}} \boldsymbol{P}_s^e, \tag{1}$$

where K_s^e, $\boldsymbol{P}_s^e$, and $\boldsymbol{v}_s^e$ are the stiffness matrix, the nodal-force vector, and the vector of the nodal unknowns of the superelement, respectively.

On each side of the RFE, we construct a one-dimensional (coarse) grid L_i (nested into the fine grid S_h) with size h_i ($i = 1, 2, 3, 4$). On the coarse grid L_i of the superelement, we construct additional functions u_i and v_i that approximate the displacements (see [6]):

$$u_i = \boldsymbol{N}_i \boldsymbol{q}_i^u, \qquad v_i = \boldsymbol{N}_i \boldsymbol{q}_i^v. \tag{2}$$

Here $\boldsymbol{N}_i$ is the vector of shape functions of the grid L_i, and $\boldsymbol{q}_i^u$ and $\boldsymbol{q}_i^v$ are the vectors of the nodal values of the functions u_i and v_i on the grid L_i ($i = 1, 2, 3, 4$), respectively.

Let $\boldsymbol{v}_h^e$ denote the vector of the nodal unknowns of a composite five-grid RFE that includes the FEM parameters of those nodes of the superelement which are the nodes of the coarse grids L_i. In Fig. 1, these nodes are shown by filled circles (10 nodes). We introduce the vector

$$\boldsymbol{q}_0^e = \{\boldsymbol{q}_1^u\, \boldsymbol{q}_2^u\, \boldsymbol{q}_3^u\, \boldsymbol{q}_4^u\, \boldsymbol{q}_1^v\, \boldsymbol{q}_2^v\, \boldsymbol{q}_3^v\, \boldsymbol{q}_4^v\}^{\mathrm{t}} \tag{3}$$

and write the following matrix relation between the vectors $\boldsymbol{v}_h^e$ and $\boldsymbol{q}_0^e$:

$$\boldsymbol{q}_0^e = B_s^e \boldsymbol{v}_h^e, \tag{4}$$

where B_s^e is a rectangular Boolean matrix.

We assume that the values of the functions u_i and v_i at the boundary nodes of the fine grid are equal to the corresponding components of the vector $\boldsymbol{v}_s^e$. Using these equalities and (2) and (3), the vector $\boldsymbol{v}_s^e$ is expressed in terms of $\boldsymbol{q}_0^e$:

$$\boldsymbol{v}_s^e = A_s^e \boldsymbol{q}_0^e, \tag{5}$$

where A_s^e is a rectangular matrix.

Substituting (5) into (1) and using (4) and the condition $\partial \Pi_s^e / \partial \boldsymbol{v}_h^e = 0$, we obtain

$$K_t^e \boldsymbol{v}_h^e = \boldsymbol{F}_t^e, \tag{6}$$

where $K_t^e = (B_s^e)^{\mathrm{t}} (A_s^e)^{\mathrm{t}} K_s^e A_s^e B_s^e$ is the stiffness matrix and $\boldsymbol{F}_t^e = (B_s^e)^{\mathrm{t}} (A_s^e)^{\mathrm{t}} \boldsymbol{P}_s^e$ is the nodal-force vector of the composite five-grid RFE.

Thus, the RFE is a five-grid finite element since it contains five nodal grids — one grid S_h and four grids L_i. For $a = b$ and $h_1 = h_2 = h_3 = h_4 = H$, we obtain a square two-grid FE. CMFEs shaped like a triangle and a rectangular parallelepiped can be constructed in a similar manner. Figure 2 shows a four-grid FE shaped like a right-angle prism, which has a three-dimensional fine grid with step size h along the axes Ox, Oy, and Oz and three two-dimensional coarse grids lying on the adjacent faces of this FE. The nodes of the coarse grids are shown by filled circles. A discrete composite model that consists of m-grid finite elements is called an m-grid model.

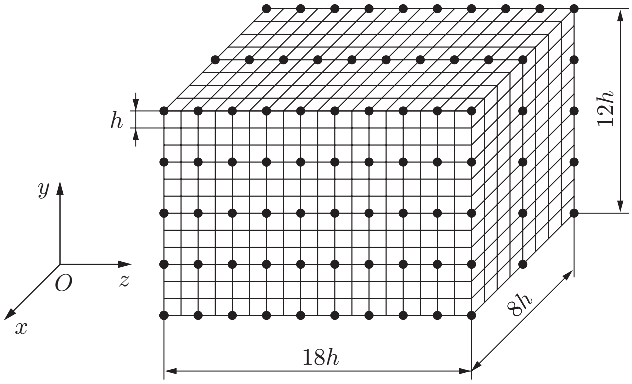

Fig. 2. Four-grid FE V_e^p.

Remark 1. We introduce the vector $\boldsymbol{W}_0$ of nodal displacements in the basic model (basic partition) of the composite. Let $\|\boldsymbol{W}^0 - \boldsymbol{W}_0\| \leqslant \delta_1$, where $\boldsymbol{W}^0$ is an exact solution and let $\|\boldsymbol{W}_0 - \boldsymbol{W}^h\| \leqslant \delta$, where $\boldsymbol{W}^h$ is the nodal displacement vector of the composite model. In this case, we obtain $\|\boldsymbol{W}^0 - \boldsymbol{W}^h\| \leqslant \delta_0$, where $\delta_0 = \delta_1 + \delta$. The error δ_1 is defined by the basic partition of the composite. The factors affecting this error have been studied in FEM theory [7]. Let $\delta_1 = 0$ for the basic partition. In this case, testing of the multigrid model reduces to determining the error δ. Calculations show that the most significant change in δ is observed for simultaneous variation in the structures (step sizes) of the fine and coarse grids.

2. Using Mean Local Errors in Multigrid Modeling of Composites. Calculations show that the error δ (see Remark 1) is a function of coordinates and its values can vary over a wide range. We note that, in calculations, it is most important to know the error in the maximum displacements and stresses. In this connection, in the analysis of grid solution, it is proposed to use the mean local errors determined for displacements (stresses) in small subregions lying at the center of the CFME. We consider the mean local errors in multigrid modeling of composites using some propositions, whose essence is illustrated for two-grid models of two-dimensional composites.

2.1. Basic Propositions of Two-Grid Models for Two-Dimensional Composites. *Proposition 1.* A two-dimensional composite located in a Cartesian coordinate system xOy is subjected to plane stresses and represented by square regions S_e with side a, where $e = 1, \ldots, N$ (N is the total number of regions S_e). The basic partition (basic model) of the composite, consisting of first-order square FEs S_j^h with side h, takes into account the structure of the composite. The components of the composite are isotropic homogeneous bodies. The two-grid model of the composite consists of two-grid square FEs S_e^p with side a ($e = 1, \ldots, N$, where N is the total number of FEs S_e^p), whose composite structures are regular and different. The basic partition of the region S_e of the two-grid FE S_e^p consists of square FEs S_j^h and generates a fine square grid S_h with step size h (as in the region S_e of the basic composite model). On the sides of the FE S_e^p there are four identical coarse grids: L_1, L_2, L_3, and L_4 with the step size $H = kh$, where k is an integer. The two-grid FE S_e^p have identical fine and one-dimensional coarse grids (S_h and L_i, respectively). Let $\boldsymbol{w}_h^e$ and $\boldsymbol{w}_0^e$ denote the nodal displacement vectors of the fine grid S_h of the region S_e that correspond to the two-grid and basic models, respectively.

Proposition 2. For the regions S_e of the basic and two-grid models of the composite, we use the same law of partition into FE S_j^h. For the region S_e, we construct a sequence of partitions $\{R_n\}_{n=1}^{n=\infty}$ (basic partitions) that take into account the composite structure of the region S_e for any n. For the partition R_n, the step size h of the fine grid S_h is given by $h = h_0/n$, where $h_0 = a/l$ (l is an integer); in this case, we have $k_1 = l/k$, where k_1 is an integer. The notation $h \to 0$ means that $h = h_0/n \to 0$ as $n \to \infty$, where $a, h_0, k = \text{const}$; i.e., the quantities a, l, and k are specified. It is worth noting that, given the partition law for the region S_e, the square FEs S_j^h of the partitions R_n are isotropic and homogeneous. It is well known [8] that the stiffness-matrix coefficients of the first-order isotropic homogeneous square FEs S_j^h with side h are independent of h and limited. Hence, as $h \to 0$, the coefficients of the finite-element equations constructed for the partitions R_n do not increase; i.e., they are limited.

Proposition 3. We write the vectors $\boldsymbol{w}_h^e$ and $\boldsymbol{w}_0^e$ in the form

$$\boldsymbol{w}_0^e = \{\boldsymbol{u}_0^e\, \boldsymbol{v}_0^e\}^{\mathrm{t}}, \qquad \boldsymbol{w}_h^e = \{\boldsymbol{u}_h^e\, \boldsymbol{v}_h^e\}^{\mathrm{t}}, \tag{7}$$

where $\boldsymbol{v}_0^e(\boldsymbol{v}_h^e)$ is a vector that contain the displacement values of all nodes of the coarse grids L_i of the region S_e

and $\boldsymbol{u}_0^e(\boldsymbol{u}_h^e)$ is a vector that contains the displacement values of the nodes of the fine grid S_h of the region S_e that do not coincide with the nodes of the coarse grids.

We assume that $\|\boldsymbol{w}_0^e - \boldsymbol{w}_h^e\| \to 0$ $(e = 1, \ldots, N)$ as $h \to 0$ for two-grid square FEs of any regular composite structure; i.e., let

$$h \to 0: \qquad \|\boldsymbol{u}_0^e - \boldsymbol{u}_h^e\| \to 0, \quad \|\boldsymbol{v}_0^e - \boldsymbol{v}_h^e\| \to 0, \qquad \|\boldsymbol{u}\| = \max |u_i|, \tag{8}$$

where u_i are the components of the vector $\boldsymbol{u}$.

Let the basic partition of the composite be such that the finite-element solution can be considered exact, i.e., we assume that the displacement vectors $\boldsymbol{w}_0^e$ (vectors $\boldsymbol{u}_0^e$ and $\boldsymbol{v}_0^e$) are an exact solution.

Proposition 4. The mean local errors are determined for the grid displacements and stresses in a small region S_r^{qe} at the center of a square two-grid FE S_e^p $(S_r^{qe} \subset S_e)$ and contains q nodes of the fine grid such that $2q$ is the dimension of the vector $\boldsymbol{v}_0^e$. We assume that for any h in the region S_r^{qe}, the displacements, stresses, and equivalent stresses are limited and nonzero. With allowance for (7), the finite-element system of equations for the partition S_h of the region S_e of the basic composite model can be written in matrix form

$$\begin{bmatrix} A_0^e & B_0^e \\ C_0^e & D_0^e \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{u}_0^e \\ \boldsymbol{v}_0^e \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{R}_0^e \\ \boldsymbol{P}_0^e \end{array} \right\}, \qquad K_0^e = \begin{bmatrix} A_0^e & B_0^e \\ C_0^e & D_0^e \end{bmatrix}, \qquad \boldsymbol{F}_0^e = \left\{ \begin{array}{c} \boldsymbol{R}_0^e \\ \boldsymbol{P}_0^e \end{array} \right\}, \tag{9}$$

where A_0^e and D_0^e are square matrices, B_0^e and C_0^e are rectangular matrices, $\boldsymbol{R}_0^e$ is the vector of nodal forces acting at the nodes of the fine grid S_h that do not coincide with the coarse-grid nodes, $\boldsymbol{P}_0^e$ is the vector of nodal forces acting at the nodes of the coarse grids in the region S_e, K_0^e is the stiffness matrix, and $\boldsymbol{F}_0^e$ is the vector of nodal forces of the partition S_h. The displacement vectors $\boldsymbol{u}_0^e$ and $\boldsymbol{v}_0^e$ take into account the boundary conditions of the region S_e, and the dimension of the vector $\boldsymbol{u}_0^e$ is greater than that of the vector $\boldsymbol{v}_0^e$.

From system (9), we obtain $\boldsymbol{u}_0^e = E_0^e \boldsymbol{v}_0^e$, where $E_0^e = (A_0^e)^{-1}\boldsymbol{R}_0^e - (A_0^e)^{-1}B_0^e$ and $(A_0^e)^{-1}$ is the inverse matrix. We us $\boldsymbol{u}_0^{qe}$ to denote the nodal-displacement vector of the region S_r^{qe} that corresponds to equilibrium of the basic model. Let the number of nodes q in the region S_r^{qe} be such that the dimensions of the vectors $\boldsymbol{u}_0^{qe}$ and $\boldsymbol{v}_0^e$ are equal to $2q$. Using the matrix E_0^e and taking into account that $\boldsymbol{u}_0^{qe} \subset \boldsymbol{u}_0^e$, we construct the equality $\boldsymbol{u}_0^{qe} = (A_0^{qe}\boldsymbol{R}_0^e - Q_0^{qe})\boldsymbol{v}_0^e$, where A_0^{qe} and Q_0^{qe} are rectangular and square matrices, respectively. Let $\boldsymbol{R}_0^e = \{\boldsymbol{R}_p^e \, \boldsymbol{R}_g^e\}^{\mathrm{t}}$, where $\boldsymbol{R}_g^e$ is the vector of nodal forces acting on the boundary of the region S_e (we note that since the vector $\boldsymbol{w}_0^e$ is unknown, the forces $\boldsymbol{R}_g^e$ are also unknown) and $\boldsymbol{R}_p^e$ is the vector of nodal forces acting inside the region S_e, i.e., the vector of specified nodal forces. It is well known that the farther the point of application of a point force from the region S_r^{qe}, the smaller its effect on the displacement field in this region. We assume that the forces $\boldsymbol{R}_g^e$ have little effect on the displacements in the region S_r^{qe}, i.e., the displacements are determined under the assumption that $\boldsymbol{R}_g^e = 0$. We note that $\boldsymbol{R}_g^e$ is the part of the nodal forces of the region S_e that are distributed uniformly along the region boundary. We find the displacements $\boldsymbol{u}_p^{qe} = (A_0^{qe}\{\boldsymbol{R}_p^e\, 0\}^{\mathrm{t}} - Q_0^{qe})\boldsymbol{v}_0^e$. Let $\varepsilon_0^e = \|\boldsymbol{u}_0^{qe} - \boldsymbol{u}_p^{qe}\|$ be a small quantity such that we can set $\varepsilon_0^e = 0$, i.e., $\boldsymbol{u}_0^{qe} = \boldsymbol{u}_p^{qe}$. In this case, the nodal displacements $\boldsymbol{u}_0^{qe}$ of the region S_r^{qe} are calculated by the formula

$$\boldsymbol{u}_0^{qe} = G_0^{qe}\boldsymbol{v}_0^e, \tag{10}$$

where $G_0^{qe} = A_0^{qe}\{\boldsymbol{R}_p^e\, 0\}^{\mathrm{t}} - Q_0^{qe}$ is a square matrix; the nodal-force vector $\boldsymbol{R}_p^e$ is specified.

The relations between the parameters a, h, and k of the two-grid FEs S_e^p, for which the representation (10) is used with a specified error ε_0^e, are determined from results of numerical experiments. Propositions similar to Propositions 1–4 are also formulated for two-grid (four-grid) models of three-dimensional composites consisting of two-grid (four-grid) FEs shaped like a cube (rectangular parallelepiped) whose composite structures are regular and different.

2.2. Procedure for Constructing Two-Grid Composite Models with a Specified Local Error in Displacements. We consider the main principles of this procedure using as an example a two-grid model for a two-dimensional composite. The model consists of two-grid FEs S_e^p and satisfies Propositions 1–4. Let a grid solution be constructed for this model; i.e., let the vectors $\boldsymbol{v}_h^e$ $(e = 1, \ldots, N)$ be determined. We note that $\boldsymbol{v}_h^e$ is the nodal-displacement vector of the two-grid FE S_e^p.

We write the vector $\boldsymbol{u}_h^e$ [see formula (7)] in the form $\boldsymbol{u}_h^e = \{\boldsymbol{u}_s^e \, \boldsymbol{v}_g^e\}^{\mathrm{t}}$, where $\boldsymbol{u}_s^e$ is a vector that contains the displacement values of the internal nodes of the fine grid S_h in the region S_e and $\boldsymbol{v}_g^e$ is a vector that contains the displacement values of the boundary nodes of the grid S_h that do not coincide with the coarse-grid nodes. In this case, the vector $\boldsymbol{v}_s^e$ of boundary nodal displacements of the grid S_h (i.e., the nodal-displacement vector for the superelement constructed for the partition S_h of the region S_e) has the form $\boldsymbol{v}_s^e = \{\boldsymbol{v}_g^e\,\boldsymbol{v}_h^e\}^{\mathrm{t}}$. Using the matrix relations for the superelement, we express the vector $\boldsymbol{u}_s^e$ in terms of $\boldsymbol{v}_s^e$ (see [2]):

$$\boldsymbol{u}_s^e = M_s^e \boldsymbol{v}_s^e. \tag{11}$$

Here M_s^e is a rectangular matrix.

Let $a = b$ and $h_1 = h_2 = h_3 = h_4 = H$. Substitution of (4) and (5) into (11) yields $\boldsymbol{u}_s^e = E_s^e \boldsymbol{v}_h^e$, where $E_s^e = M_s^e A_s^e B_s^e$. We introduce the vector $\boldsymbol{u}_h^{qe}$ of the nodal displacements of the region S_r^{qe} that corresponds to equilibrium of the two-grid model. Using the matrix E_s^e and taking into account that $\boldsymbol{u}_h^{qe} \subset \boldsymbol{u}_s^e$, we obtain

$$\boldsymbol{u}_h^{qe} = G_h^{qe} \boldsymbol{v}_h^e, \tag{12}$$

where G_h^{qe} is a square matrix.

For the region S_r^{qe}, we calculate the vector $\tilde{u}_h^{qe}$ by the formula

$$\tilde{u}_h^{qe} = G_0^{qe} \boldsymbol{v}_h^e. \tag{13}$$

Combining (10) and (13), we obtain the inequality

$$\|\boldsymbol{u}_0^{qe} - \tilde{\boldsymbol{u}}_h^{qe}\| \leqslant \|G_0^{qe}\|\,\|\boldsymbol{v}_0^e - \boldsymbol{v}_h^e\|. \tag{14}$$

As $h \to 0$, the coefficients of the matrix K_0^e appearing in (9) are limited (see Proposition 2) and, hence, the coefficients of the matrix G_0^{qe} are limited. Therefore, the norm of the square matrix G_0^{qe} is limited as $h \to 0$ [9]. Consequently, there exists a quantity $C_e > 0$ such that

$$\|G_0^{qe}\| \leqslant C_e < \infty \qquad (e = 1, \ldots, N) \tag{15}$$

as $h \to 0$. Since $\boldsymbol{u}_0^{qe} \subset \boldsymbol{u}_0^e$ and $\boldsymbol{u}_h^{qe} \subset \boldsymbol{u}_s^e \subset \boldsymbol{u}_h^e$, relation (8) implies

$$\|\boldsymbol{u}_0^{qe} - \boldsymbol{u}_h^{qe}\| \to 0 \qquad \text{as} \quad h \to 0. \tag{16}$$

Using (15) and (8), from inequality (14) we obtain

$$\|\boldsymbol{u}_0^{qe} - \tilde{\boldsymbol{u}}_h^{qe}\| \to 0 \qquad \text{as} \quad h \to 0. \tag{17}$$

The inequality

$$\|\tilde{\boldsymbol{u}}_h^{qe} - \boldsymbol{u}_h^{qe}\| \leqslant \|\tilde{\boldsymbol{u}}_h^{qe} - \boldsymbol{u}_0^{qe}\| + \|\boldsymbol{u}_0^{qe} - \boldsymbol{u}_h^{qe}\|$$

can be combined with (16) and (17) to give

$$\|\tilde{\boldsymbol{u}}_h^{qe} - \boldsymbol{u}_h^{qe}\| \to 0 \qquad \text{as} \quad h \to 0. \tag{18}$$

For the grid displacements of the region S_r^{qe}, we define the mean local (relative) error ε_u^e and the quantity δ_u^e as follows:

$$\varepsilon_u^e = \frac{1}{2q} \sum_{j=1}^{2q} \Big| \frac{u_{0j}^{qe} - u_{hj}^{qe}}{u_{0j}^{qe}} \Big|, \qquad \delta_u^e = \frac{1}{2q} \sum_{j=1}^{2q} \Big| \frac{\tilde{u}_{hj}^{qe} - u_{hj}^{qe}}{u_{hj}^{qe}} \Big|, \qquad e = 1, \ldots, N. \tag{19}$$

Here u_{0j}^{qe}, u_{hj}^{qe}, and $\tilde{u}_{hj}^{qe}$ are the components of the vectors $\boldsymbol{u}_0^{qe}$, $\boldsymbol{u}_h^{qe}$, and $\tilde{\boldsymbol{u}}_h^{qe}$, respectively, $2q$ is the dimension of these vectors, and q is the total number of nodes in the region S_r^{qe}.

According to (19), we have $\varepsilon_u^e = \varepsilon_u^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe})$, and $\delta_u^e = \delta_u^e(\tilde{\boldsymbol{u}}_h^{qe}, \boldsymbol{u}_h^{qe})$. By virtue of (16), (18), and (19) and since convergence in the norm (8) is equivalent to uniform convergence (i.e., $|u_{01}^{qe} - u_{h1}^{qe}| \to 0, \ldots, |\tilde{u}_{h2q}^{qe} - u_{h2q}^{qe}| \to 0$ as $h \to 0$) and the displacements in S_r^{qe} are limited and nonzero (see Proposition 4), we obtain

$$\text{as} \quad h \to 0\colon \qquad \varepsilon_u^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe}) \to 0, \quad \delta_u^e(\tilde{\boldsymbol{u}}_h^{qe}, \boldsymbol{u}_h^{qe}) \to 0. \tag{20}$$

By virtue of (20), for any $\varepsilon_0^r > 0$, there exists h or there exist vectors $\boldsymbol{u}_h^{qe}$ and $\tilde{\boldsymbol{u}}_h^{qe}$ ($\boldsymbol{u}_0^{qe} = \text{const}$ because $\boldsymbol{u}_0^{qe}$ is an exact solution, see Proposition 3) such that

$$\varepsilon_u^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe}) < \varepsilon_0^r, \qquad \delta_u^e(\tilde{\boldsymbol{u}}_h^{qe}, \boldsymbol{u}_h^{qe}) < \varepsilon_0^r. \tag{21}$$

Let ε_0^r be a small quantity such that $\varepsilon_u^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe})$ and $\delta_u^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe})$ can be considered equal, i.e.,

$$\varepsilon_u^e = \delta_u^e. \tag{22}$$

Then, by virtue of (21) and (22), we infer that if $\delta_u^e < \varepsilon_0^r$, then the estimate for ε_u^e is given by

$$\varepsilon_u^e < \varepsilon_0^r. \tag{23}$$

In the two-grid model of a composite, we distinguish a set of regions S_r^{qe} (i.e., the region consisting of two-grid FEs S_e^p) in which the grid displacements (u or v) are maximal (in magnitude). Using formulas (12), (13), and (19), for this set of regions S_r^{qe} we find the values of δ_u^e, where $e = 1, \ldots, N_1$ ($N_1 < N$); N_1 is the number of chosen regions S_r^{qe} (number of chosen FEs S_e^p). If $\delta_u^e \geqslant \varepsilon_0^r$ (where the constant ε_0^r is specified) for the chosen region S_r^{qe}, we diminish the step size h of the basic partitions of all two-grid FEs of the composite by virtue of (20) (the step size h is varied according to the rule of Proposition 2) and find a solution for the newly constructed two-grid model. As a result, we obtain a two-grid model such that the conditions $\delta_u^e < \varepsilon_0^r$ (i.e., $\varepsilon_u^e < \varepsilon_0^r$), where $e = 1, \ldots, N_1$ hold for all chosen regions. Thus, in the chosen regions S_r^{qe} in the two-grid model constructed, the mean local error ε_u^e is smaller than the specified estimate ε_0^r. In (23), it is expedient to use the values $\varepsilon_0^r \leqslant 0.01$ (i.e., $\varepsilon_0^r \leqslant 1\%$). If the displacement functions vary only slightly on S_e, the estimate for ε_0^r can be extended to the entire region S_e.

2.3. Procedure for Constructing Two-Grid Models for Specified Mean Local Error in Stresses. We consider the main principles of this procedure using as an example a two-grid model for a two-dimensional composite that consists of two-grid square FEs S_e^p and satisfies Propositions 1–4, where $e = 1, \ldots, N$ (N is the total number of FEs S_e^p). Let s solution (i.e., vectors $\boldsymbol{v}_h^e$) be constructed for the two-grid model.

We introduce the vectors $\boldsymbol{u}_0^{je}$, $\boldsymbol{u}_h^{je}$, and $\tilde{\boldsymbol{u}}_h^{je}$ that contain the nodal displacements of the jth square FE S_j^h of the region S_r^{qe} and correspond to the nodal-displacement vectors $\boldsymbol{u}_0^{qe}$, $\boldsymbol{u}_h^{qe}$, and $\tilde{\boldsymbol{u}}_h^{qe}$ ($j = 1, \ldots, m$, where m is the total number of FEs S_j^h of the region S_r^{qe}). Let $\boldsymbol{t}_0^{je} = \{\sigma_x^{0j}\, \sigma_y^{0j}\, \tau_{xy}^{0j}\}^{\mathrm{t}}$, $\boldsymbol{t}_h^{je} = \{\sigma_x^{hj}\, \sigma_y^{hj}\, \tau_{xy}^{hj}\}^{\mathrm{t}}$, and $\tilde{\boldsymbol{t}}_h^{je} = \{\tilde{\sigma}_x^{hj}\, \tilde{\sigma}_y^{hj}\, \tilde{\tau}_{xy}^{hj}\}^{\mathrm{t}}$ be the vectors of stresses $\sigma_x^{0j}, \ldots, \tilde{\tau}_{xy}^{hj}$ at the center of gravity of the FE S_j^h that correspond to the displacement vectors $\boldsymbol{u}_0^{je}$, $\boldsymbol{u}_h^{je}$, and $\tilde{\boldsymbol{u}}_h^{je}$, respectively. Since the region S_e in the two-grid and basic models consists of square FE S_j^h with side h (see Propositions 1 and 2), the basic functions of the FE S_j^h of the region S_r^{qe} are equal for the two-grid and basic models. Consequently, the vectors $\boldsymbol{t}_0^{je}$, $\boldsymbol{t}_h^{je}$, and $\tilde{\boldsymbol{t}}_h^{je}$ can be written as $\boldsymbol{t}_0^{je} = D_j^e \boldsymbol{u}_0^{je}$, $\boldsymbol{t}_h^{je} = D_j^e \boldsymbol{u}_h^{je}$, and $\tilde{\boldsymbol{t}}_h^{je} = D_j^e \tilde{\boldsymbol{u}}_h^{je}$, where D_j^e is a rectangular matrix. For simplicity, we take into account that $\boldsymbol{u}_0^{je} \subset \boldsymbol{u}_0^{qe}$, $\boldsymbol{u}_h^{je} \subset \boldsymbol{u}_h^{qe}$, and $\tilde{\boldsymbol{u}}_h^{je} \subset \tilde{\boldsymbol{u}}_h^{qe}$ and write the vectors $\boldsymbol{t}_0^{je}$, $\boldsymbol{t}_h^{je}$, and $\tilde{\boldsymbol{t}}_h^{je}$ in the form

$$\boldsymbol{t}_0^{je} = M_j^e \boldsymbol{u}_0^{qe}, \qquad \boldsymbol{t}_h^{je} = M_j^e \boldsymbol{u}_h^{qe}, \qquad \tilde{\boldsymbol{t}}_h^{je} = M_j^e \tilde{\boldsymbol{u}}_h^{qe}. \tag{24}$$

Here M_j^e ($j = 1, \ldots,$) is a rectangular matrix.

The coefficients $M_{\alpha\beta}^{je}$ of the matrix M_j^e depend on the elastic moduli of the square element S_j^h (with side h) and the partial derivatives of its basic functions limited within the FE S_j^h for any h. It follows that $|M_{\alpha\beta}^{je}| < \infty$ as $h \to 0$. For the grid equivalent stresses in the region S_r^{qe}, we use the mean local (relative) error ε_σ^e and quantity δ_σ^e calculated by the formulas

$$\varepsilon_\sigma^e = \frac{1}{m} \sum_{j=1}^{m} \Big| \frac{\sigma_{0j}^{qe} - \sigma_{hj}^{qe}}{\sigma_{0i}^{qe}} \Big|, \qquad \delta_\sigma^e = \frac{1}{m} \sum_{j=1}^{m} \Big| \frac{\sigma_{hj}^{qe} - \tilde{\sigma}_{hj}^{qe}}{\sigma_{hj}^{qe}} \Big|, \qquad e = 1, \ldots, N. \tag{25}$$

We determine the equivalent stresses σ_{0j}^{qe}, σ_{hj}^{qe}, and $\tilde{\sigma}_{hj}^{qe}$ at the center of the jth FE S_j^h according to the fourth theory of strength, i.e., we use the relation

$$\sigma_{0j}^{qe} = \sqrt{(\sigma_x^{0j})^2 + (\sigma_y^{0j})^2 - \sigma_x^{0j}\sigma_y^{0j} + 3(\tau_{xy}^{0j})^2}, \qquad \boldsymbol{t}_0^{je} = \{\sigma_x^{0j}\, \sigma_y^{0j}\, \tau_{xy}^{0j}\}^{\mathrm{t}}, \tag{26}$$

the values of σ_{hj}^{qe} and $\tilde{\sigma}_{hj}^{qe}$ are calculated by formula (26) in which the stress-vector components $\boldsymbol{t}_0^{je}$ are replaced by the corresponding components of the stress vectors $\boldsymbol{t}_h^{je}$ or $\tilde{\boldsymbol{t}}_h^{je}$.

According to (24) and (25), we obtain $\varepsilon_\sigma^e = \varepsilon_\sigma^e(\boldsymbol{u}_0^{qe}, \boldsymbol{u}_h^{qe})$ and $\delta_\sigma^e = \delta_\sigma^e(\tilde{\boldsymbol{u}}_h^{qe}, \boldsymbol{u}_h^{qe})$. By virtue of (12) and (13), we have $\boldsymbol{u}_h^{qe} = G_h^{qe} \boldsymbol{v}_h^e$ and $\tilde{\boldsymbol{u}}_h^{qe} = G_0^{qe} \boldsymbol{v}_h^e$ and, hence, $\varepsilon_\sigma^e = \varepsilon_\sigma^e(\boldsymbol{u}_0^{qe}, G_h^{qe} \boldsymbol{v}_h^e)$ and $\delta_\sigma^e = \delta_\sigma^e(G_0^{qe} \boldsymbol{v}_h^e, G_h^{qe} \boldsymbol{v}_h^e)$. Since $\boldsymbol{u}_0^{qe} = \text{const}$ ($\boldsymbol{u}_0^{je} \subset \boldsymbol{u}_0^{qe} \subset \boldsymbol{u}_0^e$ is an exact solution, see Proposition 3), we obtain

$$\varepsilon_\sigma^e = \varepsilon_\sigma^e(\boldsymbol{v}_h^e), \qquad \delta_\sigma^e = \delta_\sigma^e(\boldsymbol{v}_h^e). \tag{27}$$

Consequently, the functions $y_e = y_e(\boldsymbol{v}_h^e)$ and $x_e = x_e(\boldsymbol{v}_h^e)$, where

$$y_e = \varepsilon_\sigma^e(\boldsymbol{v}_h^e), \qquad x_e = \delta_\sigma^e(\boldsymbol{v}_h^e), \tag{28}$$

are related by an equation of the form $y_e = F_e(x_e)$. Using (16), (18), and (24) and taking into account that $|M^{je}_{\alpha\beta}| < \infty$ and the norm (8) ensures uniform convergence $\boldsymbol{u}^{qe}_h \to \boldsymbol{u}^{qe}_0$ and $\tilde{\boldsymbol{u}}^{qe}_h \to \boldsymbol{u}^{qe}_h$ as $h \to 0$, one can easily show that

$$\|\boldsymbol{t}^{je}_h - \boldsymbol{t}^{je}_0\| = \|M^e_j(\boldsymbol{u}^{qe}_h - \boldsymbol{u}^{qe}_0)\| \to 0, \qquad \|\tilde{\boldsymbol{t}}^{je}_h - \boldsymbol{t}^{je}_h\| = \|M^e_j(\tilde{\boldsymbol{u}}^{qe}_h - \boldsymbol{u}^{qe}_0)\| \to 0$$

as $h \to 0$. Using (26) and taking into account that the uniform convergence $\boldsymbol{t}^{je}_h \to \boldsymbol{t}^{je}_0$ and $\tilde{\boldsymbol{t}}^{je}_h \to \boldsymbol{t}^{je}_h$ (i.e., $|\sigma^{0j}_x - \sigma^{hj}_x| \to 0$, $|\sigma^{0j}_y - \sigma^{hj}_y| \to 0$, ..., and $|\tilde{\sigma}^{hj}_x - \sigma^{hj}_x| \to 0$ as $h \to 0$) and $\sigma^{qe}_{0j}, \sigma^{qe}_{hj}, \tilde{\sigma}^{qe}_{hj} < \infty$ and $\sigma^{qe}_{0j}, \sigma^{qe}_{hj}, \tilde{\sigma}^{qe}_{hj} \neq 0$ (see Proposition 4), one can readily show that $|\sigma^{qe}_{hj} - \sigma^{qe}_{0j}| \to 0$ and $|\tilde{\sigma}^{qe}_{hj} - \sigma^{qe}_{hj}| \to 0$ as $h \to 0$. By virtue of (25), we obtain

$$h \to 0: \qquad \varepsilon^e_\sigma \to 0, \quad \delta^e_\sigma \to 0. \tag{29}$$

Using (8), (27), and (29), we infer that $\varepsilon^e_\sigma(\boldsymbol{v}^e_h) \to \varepsilon^e_\sigma(\boldsymbol{v}^e_0) = 0$ and $\delta^e_\sigma(\boldsymbol{v}^e_h) \to \delta^e_\sigma(\boldsymbol{v}^e_0) = 0$ as $h \to 0$, i.e., as $\boldsymbol{v}^e_h \to \boldsymbol{v}^e_0$. From this, by virtue of (28) we have $y_e(\boldsymbol{v}^e_h) \to y_e(\boldsymbol{v}^e_0) = 0$ and $x_e(\boldsymbol{v}^e_h) \to x_e(\boldsymbol{v}^e_0) = 0$ for $\boldsymbol{v}^e_h \to \boldsymbol{v}^e_0$. Thus, the functions $y_e(\boldsymbol{v}^e_h)$ and $x_e(\boldsymbol{v}^e_h)$ vanish at the same point ($\boldsymbol{v}^e_0$) and, hence, $F_e(0) = 0$ on the right since $x_e \geqslant 0$.

Similar reasoning to that in Sec. 2.2 shows that if $\delta^e_\sigma < \delta^r_0$, the error ε^e_σ has an estimate $\varepsilon^e_\sigma < \delta^r_0$, where δ^r_0 is specified. Let δ^r_0 be a small quantity such that $\delta^e_\sigma = \varepsilon^e_\sigma$. These conditions for the error ε^e_σ hold for small values of δ^r_0 ($\delta^r_0 \leqslant 0.01$) and, hence, they are difficult to satisfy since it is necessary to use very fine basic partitions of FE S^p_e. We consider another method for estimating the errors ε^e_σ.

Since $F_e(0) = 0$, the function $F_e(x_e)$ in the ε-neighborhood of the zero point (on the right of zero), i.e., on the segment $[0, \varepsilon]$ can be written in approximate form $F_e(x_e) = b_e x_e + a^e_1 x^2_e + \ldots + a^e_n x^{n+1}_e$, where $b_e, a^e_i = \text{const}$, n is an integer, and $0 \leqslant x_e \leqslant \varepsilon$. Let $\varepsilon \ll 1$, i.e., $x_e \ll 1$. Setting $a^e_1 x^2_e = 0, \ldots, a^e_n x^{n+1}_e = 0$, we obtain $F_e(x_e) = b_e x_e$, i.e., a linear function of the form $y_e = b_e x_e$, where $b_e > 0$ since $x_e y_e \geqslant 0$. For this linear function and any value of δ_r ($0 < \delta_r < \varepsilon$), we find that if $x_e < \delta_r$, then $y_e < \varepsilon^e_r$, where $\varepsilon^e_r = b_e \delta_r$. With allowance for (28), it follows that if $\delta^e_\sigma < \delta_r$, the following estimate for ε^e_σ holds:

$$\varepsilon^e_\sigma \leqslant \varepsilon_r, \qquad e = 1, \ldots, N_2. \tag{30}$$

Here $\varepsilon_r = \max(\varepsilon^e_r)$ ($e = 1, \ldots, N_2$, where N_2 is the total number of chosen regions S^{qe}_r, i.e., the number of chosen FEs S^p_e in which the stresses are analyzed, $N_2 < N$), $\varepsilon_r, \delta_r = \text{const}$, and $\varepsilon_r, \delta_r \ll 1$ (ε_r is specified).

Since $\varepsilon^e_r = b_e \delta_r$ ($b_e = \text{const}$) and $\varepsilon_r = \max(\varepsilon^e_r)$, then $\varepsilon_r = \max(b_e \delta_r)$, i.e., $\varepsilon_r = b\delta_r$, where $b = \max(b_e)$, $e = 1, \ldots, N_2$. Thus, ε_r depends on δ_r and, what is more important, the quantity ε_r decreases with δ_r. Given the quantity ε_r, we determine δ_r using test calculations. In the two-grid model of a two-dimensional composite, we choose a set of regions S^{qe}_r (i.e., subregion consisting of FEs S^p_e) in which the equivalent stresses attain the maximum value. For the regions S^{qe}_r, we determine the quantities δ^e_σ ($e = 1, \ldots, N_2$) using formula (25). If $\delta^e_\sigma \geqslant \delta_r$ for the chosen region S^{qe}_r, we decrease the step size h of the basic partitions of all two-grid FEs by virtue of (29) and find the solution for the newly constructed two-grid model. As a result, we obtain a two-grid model for which the condition $\delta^e_\sigma < \delta_r$, i.e., $\varepsilon^e_\sigma \leqslant \varepsilon_r$ is satisfied in the chosen subregion. According to calculations, in (30) for the specified $\varepsilon_r = 0.02$ ($\varepsilon_r = 2\%$), it is expedient to use values $\delta_r \leqslant 0.06$ (i.e., $\delta_r \leqslant 6\%$). If the stresses vary only slightly in S_e, the estimate of ε_r can be extended to the entire region S_e. In practice, one should use the region $S^e_r \subset S^{qe}_r$ whose shape is convenient for calculations.

Remark 2. Calculations show that for specified ε^r_0, δ_r, and $\varepsilon_r \ll 1$, estimates (23) and (30) for the errors ε^e_u and ε^e_σ, respectively, are also valid for the regions S^{qe}_r in the chosen FEs S^p_e whose structures differ. Hence, the quantities ε^r_0 and δ_r, and ε_r do no depend on the structure of the FE S^p_e. Indeed, estimates (23) and (30) are based on conditions (8) and the assumption of Proposition 3, which hold for two-grid square FEs S^p_e of any regular composite structure.

Remark 3. Similar reasoning to that made above for composite models consisting of two-grid (four-grid) FEs V^p_e shaped like a cube (right-angle prism) using propositions similar to Propositions 1–4 leads to estimates of the form (23) and (30) for the mean local errors in displacements and equivalent stresses in the subregions located at the center of the FEs V^p_e.

3. Numerical Results. We consider plane stresses in a two-dimensional composite S of irregular structure with a filling ratio to 0.218 in a Cartesian coordinate system xOy (Fig. 3a). The boundary conditions have the form $u = v = 0$ for $y = 0$ and $2a \leqslant x \leqslant 5a$ and $x = 0$ and $3a \leqslant y \leqslant 5a$ (in Fig. 3a, the fixed boundary of S is dashed). The region occupied by the composite is divided into square subregions S_e with side $a = 60h$, where $e = 1, \ldots, 43$. The composite structure consists of four typical composite square regions S^k with side a (Fig. 3b, $k = 1, 2, 3, 4$).

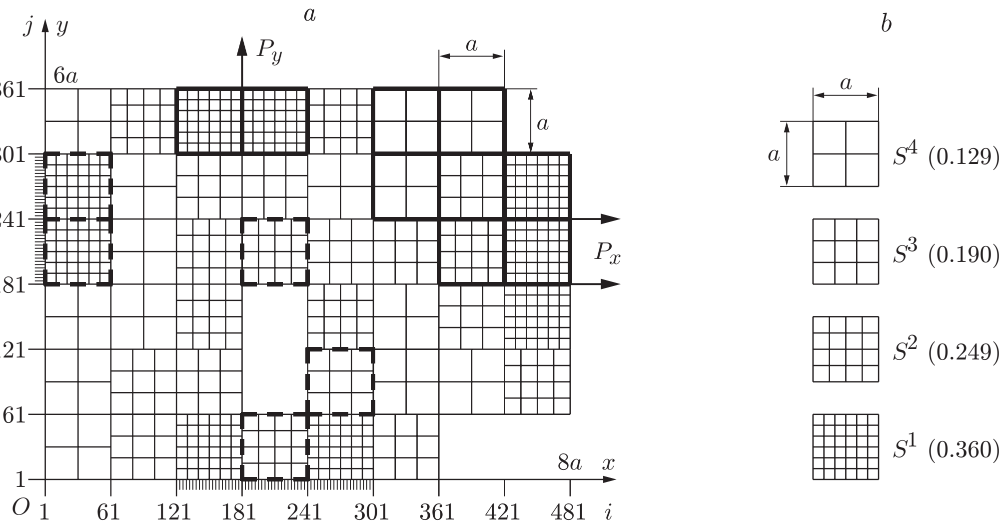

Fig. 3. Calculated diagram of the composite S (a) and composite structures of typical regions S^k (b).

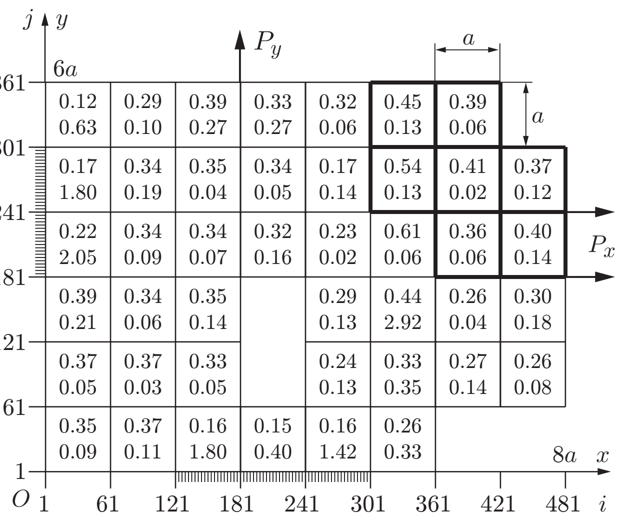

Fig. 4. Diagram of the errors ε_u^e and values of δ_u^e for the displacements of the regions $S_r^e \subset S_e$.

The region S^k is reinforced by flat fibers of width $2h$ inside the region and h on its boundary. In Fig. 3, the fibers are shown by lines and the value of the filling ratio S^k is given in parentheses. To construct a two-grid FE S_e^p in the region S_e, we use the grid of its basic partition, which consists (as the basic composite model) of first-order square FEs S^h with side h [7] and four identical one-dimensional grids with step size $4h$. For the nodes of the basic partition of the composite S, we introduce integer coordinates i and j (Fig. 3a). The forces $P_x = 87.5$ and $P_y = 96.3$ act at the nodes with coordinates $(481, 181)$, $(481, 241)$, and $(181, 361)$. Calculations were performed for $h = 0.5$, Poisson's ratio for all composite components of 0.3, a fiber Young's modulus of 10, and a binder Young's modulus of 1.

The maximum value of the displacements u_h and v_h for the two-grid model differs from the displacements u_0 and v_0 for the basic model by 0.4%. The maximum equivalent stress σ_h calculated at the center of FE S^h for the two-grid model using the fourth theory of strength [10] differs from the stresses σ_0 of the basic model by 0.3%.

Figures 4 and 5 show diagrams of the errors ε_u^e and ε_σ^e and values of δ_u^e and δ_σ^e (in percent) calculated by formulas (19) and (25) for the $4h \times 4h$ region S_r^e ($S_r^e \subset S_r^{qe}$); in the region S_e, the upper numbers refer to ε_u^e and ε_σ^e and the lower numbers to δ_u^e and δ_σ^e. For example, $\varepsilon_u^e = 0.35\%$ and $\delta_u^e = 0.09\%$ for S_e adjacent to the origin

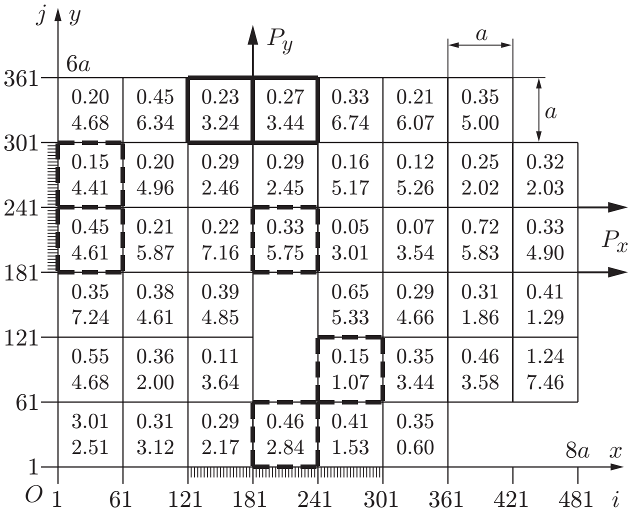

Fig. 5. Chart of the errors ε_σ^e and values of δ_σ^e for the stresses of the regions $S_r^e \subset S_e$.

of the coordinates xOy (see Fig. 4). In Figs. 3a, 4, and 5, the thick solid lines show FEs S_e^p in which σ_h or u_h and v_h attain the maximum values. An analysis shows that for all regions S_r^e of the chosen FE S_e^p, estimate (23) for ε_u^e holds for $\varepsilon_0^r = 0.01$ ($\varepsilon_0^r = 1\%$) and estimate (30) for ε_σ^e holds for $\delta_r = 0.06$ and $\varepsilon_r = 0.02$ ($\delta_r = 6\%$ and $\varepsilon_r = 2\%$). The estimates for the errors ε_u^e and ε_σ^e can be extended to the entire region S_e. We find that $\varepsilon_u^e < 1\%$ in the neighborhood of the maximum displacement of the composite and $\varepsilon_\sigma^e < 2\%$ near the points of application of the forces. For $\delta_r \leqslant 0.06$ and $\varepsilon_r = 0.02$, estimate (30) is also valid for the subregions S_r^e of the FEs S_e^p (in Figs. 3a and 5, these FEs are shown by thick dashed lines), where σ_h is approximately 10 times lower than the maximum stress. We note that for the given values of ε_0^r, δ_r, and ε_r, estimates (23) and (30) for the errors ε_u^e and ε_σ^e, respectively, are valid for S_r^e of the chosen FEs S_e^p of different composite structure (see Fig. 3a). Finite-element implementation for the two-grid composite model is 20 times faster and requires 150 times smaller computer memory than that for the basic model.

REFERENCES

1. T. Fudzii and M. Dzako, *Fracture Mechanics of Composite Materials* [Russian translation], Mir, Moscow (1982).
2. V. A. Postnov, *Numerical Methods in Calculation of Ship Structures* [in Russian], Sudostroenie, Leningrad (1977).
3. A. D. Matveev, "Elastic Multigrid Finite Elements. Analysis of Composites with Allowance for their Structure," Krasnoyarsk (1998). Deposited at VINITI 11.06.98, No. 3219-B98.
4. A. D. Matveev, "Some Approaches for Constructing Elastic Multigrid Finite Elements," Krasnoyarsk (2000). Deposited at VINITI 11.24.00, No. 2990-B00,
5. V. I. Samul', *Fundamentals of the Theory of Elasticity and Plasticity* [in Russian], Vysshaya Shkola, Moscow (1970).
6. D. H. Norrie and G. De Vries, *An Introduction to Finite-Element Analysis*, Academic Press, New York (1978).
7. G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Engiwood Cliffs, Prentice-Hall (1973).
8. P. M. Varvak, I. M. Burzun, A. S. Gorodetskii, et al., *Finite Element Method* [in Russian], Vishcha Shkola, Kiev (1981).
9. N. S. Bakhvalov, *Numerical Methods*, Vol. 1 [in Russian], Nauka, Moscow (1973).
10. G. S. Pisarenko, A. P. Yakovlev, and V. V. Matveev, *Strength of Materials* [in Russian], Naukova Dumka, Kiev (1975).